Noncompact quantum knot invariants

T. D. Dimofte[a]

[a]Caltech 452-48, 1200 E California Blvd, Pasadena, CA 91106

We describe one avenue to the explicit calculation of partition functions of knot complements in Chern-Simons theory with noncompact gauge group $SL(2,\mathbb{C})$, following [1]. Our techniques involve geometric quantization of the moduli space of flat connections on the torus, combined with quantization of the classical A-polynomial of a knot complement. We also compare these methods to known results for compact gauge group $SU(2)$.

1. SETUP

We wish to consider Chern-Simons theory with complex, noncompact gauge group $G_{\mathbb{C}} = SL(2,\mathbb{C})$. Recall that $G_{\mathbb{C}}$ is the complexification of the compact group $G = SU(2)$. In particular, an $SL(2,\mathbb{C})$ connection $\mathcal{A}$ may be decomposed as

$$\mathcal{A} = \mathcal{A}^a T^a \,, \tag{1}$$

where T^a are antihermitian generators of $su(2)$, but the coefficients $\mathcal{A}^a$ are complex. Most generally, Chern-Simons theory with a complex gauge group on a 3-manifold M has the action [2]

$$S[\mathcal{A}] = \frac{t}{2}\, I_{CS}[\mathcal{A}] + \frac{\tilde{t}}{2}\, I_{CS}[\bar{\mathcal{A}}]\,, \tag{2}$$

where

$$I_{CS}[\mathcal{A}] = \int_M \mathrm{Tr}\big(\mathcal{A}\wedge d\mathcal{A} + \frac{2}{3}\mathcal{A}\wedge\mathcal{A}\wedge\mathcal{A}\big)\,, \tag{3}$$

and the full partition function depends nontrivially on both t and $\tilde{t}$. However, if we define the partition function $Z(M)^{(\rho)} = \int_{(\rho)} \mathcal{D}\mathcal{A}\, e^{iS}$ as a *perturbative* expansion in the background of a fixed classical solution $\mathcal{A}^{(\rho)}$, it will holomorphically factorize as

$$Z(M)^{(\rho)} = Z(M;t)^{(\rho)} Z(M;\tilde{t}\,)^{(\rho)}. \tag{4}$$

Our goal is to study the holomorphic piece $Z(M;t)^{(\rho)}$ of this perturbative partition function when M is the complement of (a small neighborhood of) a knot K in the three-sphere: $M = S^3\backslash K$. The choice ρ of classical solution is related to the choice of boundary conditions on Σ. Since the action (3) is topological, one would hope that also in the quantum theory $Z(M;t)^{(\rho)}$ calculates topological invariants of K; this is (mostly) true, as explained in [3,2].

Below, we use geometric quantization to derive the quantum Hilbert space associated to the toroidal boundary Σ, and then explain how to identify wavefunctions in this Hilbert space with the partition functions $Z^{(\rho)}(M)$ for various 3-manifolds M.

2. MODULI SPACE OF FLAT CONNECTIONS

In order to find the Hilbert space $\mathcal{H}$ associated to Σ, we consider Chern-Simons theory on a 3-manifold $\Sigma\times\mathbb{R}$, where $\mathbb{R}$ can be thought of as a time coordinate [3]. It is easy to check that the classical solutions of Chern-Simons theory are given by flat connections, satisfying

$$\mathcal{F} = \mathcal{A} + \mathcal{A}\wedge\mathcal{A} = 0\,. \tag{5}$$

Flat connections are always uniquely determined by their holonomies around nontrivial 1-cycles, modulo gauge transformations. These holonomies form a representation of π_1. Since $\mathbb{R}$ is contractible, the classical solutions of Chern-Simons theory on $\Sigma\times\mathbb{R}$ are in 1-1 correspondence with flat connections on Σ itself. Therefore, the classical moduli space of the theory is

$$\begin{aligned}\mathcal{M}_{\mathbb{C}} &= \{\mathcal{A} \text{ on } \Sigma \text{ s.t. } \mathcal{F}=0\}/\text{gauge trans.}\\ &= \mathrm{Hom}(\pi_1(\Sigma),\, G_{\mathbb{C}})/\text{conjugation.}\end{aligned} \tag{6}$$

This moduli space has been extensively studied for various gauge groups and Riemann surfaces Σ. It is a finite-dimensional hyperkähler manifold with many interesting properties (*cf.* [2,4]). For $\Sigma = T^2$ and $G_{\mathbb{C}} = SL(2,\mathbb{C})$, it is easy to describe $\mathcal{M}_{\mathbb{C}}$ explicitly. Since $\pi_1(\Sigma) = \mathbb{Z}^2$ is commutative, the holonomies of $\mathcal{A}$ can (almost always) be simultaneously diagonalized using gauge transformations. Letting $e^{\pm u}$ and $e^{\pm v}$ be the eigenvalues of $\mathcal{A}$ on the two 1-cycles of Σ, we then have

$$\mathcal{M}_{\mathbb{C}} = \{(u,v)\} = (\mathbb{C}^* \times \mathbb{C}^*)/\mathbb{Z}_2\,, \tag{7}$$

where $\mathbb{Z}_2$ is the Weyl group symmetry (a remaining conjugation) that sends $(u,v) \mapsto (-u,-v)$.

The holomorphic Chern-Simons action induces a natural symplectic structure on the space $\mathcal{M}_{\mathbb{C}}$,

$$\omega = \frac{t}{8\pi}\int_{\Sigma} \delta\mathcal{A} \wedge \delta\mathcal{A} = \frac{t}{2\pi} du \wedge dv\,. \tag{8}$$

In order to quantize $\mathcal{M}_{\mathbb{C}}$, we simply promote the Poisson brackets of (8) to a commutation relation $[u,v] = -2\pi i/t$. Note how $2\pi/t$ plays the role of Planck's constant $\hbar$. By comparison to ordinary quantum mechanics, the Hilbert space $\mathcal{H}$ (by definition the quantization of $\mathcal{M}_{\mathbb{C}}$) can be considered to consist of wavefunctions $f(u)$ on which operators $\hat{u}$ and $\hat{v}$ act as $u\cdot$ and $(2\pi i/t)\partial_u$, respectively.

3. COMPARISON TO SU(2)

The above arguments work almost the same way in the case of the compact gauge group $G = SU(2)$. However, the relevant classical moduli space is $\mathcal{M} = (S^1 \times S^1)/\mathbb{Z}_2$ because u and v must be valued in $i\mathbb{R}/2\pi i\mathbb{Z}$. Moreover, the level $k = t/2$ of the compact Chern-Simons theory must be an integer. Since a compact "momentum" $\hat{v}$ quantizes the compact "position" $\hat{u}$ (and vice versa), the Hilbert space consists of functions $f(u)$ such that $u \in i\pi\mathbb{Z}/k$ and $f(u+2\pi i) = f(u)$. Combined also with the $\mathbb{Z}_2$ Weyl symmetry imposing $f(-u) = f(u)$, we see that $\mathcal{H}$ is $(k+1)$-dimensional, consisting of values of f at $u = 0,\, i\pi/k,\, 2\pi i/k, ...,\, i\pi$.

This is completely consistent with the description of $\mathcal{H}$ as the space of conformal blocks of the $SU(2)$ WZW model at level k on the torus [3], since there are $k+1$ level-k representations of $\widehat{su}(2)$. A complete basis for $\mathcal{H}$ can also be obtained by computing the Chern-Simons partition function on a *solid* torus with a Wilson loop "colored" by the first $k+1$ representations of $SU(2)$ running through its center. An n-dimensional representation on the loop induces a holonomy $\exp(\pm i\pi(n-1)/k)$ around it.

4. A-POLYNOMIAL

To finish, let us come back to $G_{\mathbb{C}} = SL(2,\mathbb{C})$ and explain which function $f(u) \in \mathcal{H}$ corresponds to $Z(M;t)^{(\rho)}$ for a particular knot complement M. Classically, the flat connections on $\partial M = \Sigma$ which extend to flat connections on M are characterized as solutions to a polynomial

$$A_M(e^u, e^v) = 0 \tag{9}$$

in the two holonomies around $\Sigma = T^2$. Typically, one chooses e^v to correspond to the cycle of Σ that is trivial in $H_1(M)$, and e^u to the complementary cycle (*i.e.* the one linking the knot). The function $A_M(e^u, e^v)$ is called the A-polynomial of K. Every classical solution ρ corresponds to a point on the curve (9). Since A_M is a polynomial, there are a finite number of classical solutions, indexed by $\rho_i(u)$, that correspond to a particular boundary condition u. In *quantum* Chern-Simons theory, the perturbative partition function must obey

$$\hat{A}_M(e^{\hat{u}}, e^{\hat{v}}) \cdot Z(M;t)^{(\rho_i(u))} = 0 \tag{10}$$

for an appropriate quantized version $\hat{A}_M$ of A_M [1,5]. This differential equation can then be solved to find the $Z(M;t)^{(\rho_i(u))}$'s as functions of u and t.

This completes our short description of $SL(2,\mathbb{C})$ knot invariants. Many more details and alternative derivations of $Z(M;t)^{(\rho)}$ appear in [1].

REFERENCES

1. T.D. Dimofte, S. Gukov, J. Lenells, and D. Zagier, to appear.
2. E. Witten, Comm. Math. Phys. 137 (1991) 29.
3. E. Witten, Comm. Math. Phys. 121 (1989) 351.
4. A. Kapustin and E. Witten, arXiv:hep-th/0604151.
5. S. Gukov, Comm. Math. Phys. 255 (2005) 577.